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Classical BRST charge for nonlinear algebras

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Abstract

We study the construction of the classical nilpotent canonical BRST charge for the nonlinear gauge algebras where a commutator (in terms of Poisson brackets) of the constraints is a finite order polynomial of the constraints. Such a polynomial is characterized by the coefficients forming a set of higher order structure constants. Assuming the set of constraints to be linearly independent, we find the restrictions on the structure constants when the nilpotent BRST charge can be written in a simple and universal form. In the case of quadratically nonlinear algebras we find the expression for third order contribution in the ghost fields to the BRST charge without the use of any additional restrictions on the structure constants.

1 Introduction

Discovery of the BRST symmetry [1], [2] exerted a large impact upon development of gauge theories. At present, the BRST charge, corresponding to the Noether current of BRST symmetry, is one of the most efficient tools for studing the classical and quantum aspects of constrained systems (see e.g. [3]). The properties of the BRST charge, especially its nilpotency, are the base of modern quantization methods of gauge theores in both Lagrangian [4] and Hamiltonian [5] formalism (see also the reviews [6]). Namely the nilpotency of BRST charge plays a crucial role for generic description of physical state space in quantum theory of gauge fields [7]. One of the most important modern applications of the BRST charge is related to string theory and (super)conformal theories [8] (see also [9]) where the use of the BRST charge allowed one to construct a physical state space, to clarify the structure of physical spectrum and derive the critical dimensions of the models. In addition, we point out that the nilpotent BRST charge was used for constructing the action of string field theory [10] (see [11] for review).

In this paper we discuss the form of the canonical BRST charge (to be more precise, the BRST - BFV charge [5]) for a general class of gauge theories. The BRST construction is based on classical formulation of the gauge theory in phase space where the gauge theory is characterized by first class constraints $T_\alpha = T_\alpha(p,q)$ with p_i and q^i being canonically conjugate phase variables. Constraints T_α satisfy the involution relations in terms of the Poisson bracket

$$\{T_\alpha, T_\beta\} = f^{\gamma}_{\alpha\beta} T_\gamma \tag{1}$$

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with structure functions $f^{\gamma}_{\alpha\beta}(q,p)$. In Yang-Mills type theories the structure functions are constants and the nilpotent BRST charge $\mathcal{Q}$ ($\{\mathcal{Q},\mathcal{Q}\}=0$) can be written in a closed form as follows

$$\mathcal{Q} = c^{\alpha}T_{\alpha} - \frac{1}{2}c^{\alpha}c^{\beta}f^{\gamma}_{\alpha\beta}\mathcal{P}_{\gamma} \tag{2}$$

where c^{α} and $\mathcal{P}_{\alpha}$ are canonically conjugate ghost variables. For general gauge theories the structure functions depend on phase variables $f^{\gamma}_{\alpha\beta} = f^{\gamma}_{\alpha\beta}(p,q)$ and the nilpotent BRST charge is defined by series expansion (in general, infinite) in ghost variables

$$\mathcal{Q} = c^{\alpha}T_{\alpha} - \frac{1}{2}c^{\alpha}c^{\beta}f^{\gamma}_{\alpha\beta}\mathcal{P}_{\gamma} + \cdots. \tag{3}$$

Here the dots mean the terms of higher orders in ghost variables [5] conditioned by p,q dependence of structure functions. In general, a closed form for this series is unknown.

Development of conformal field theories led to discovery of a new class of gauge theories possessing the nonlinear gauge algebras, called $\mathcal{W}_N$ algebras, where structure functions essentially depend on the phase variables (see [12] for $\mathcal{W}_3$ algebra and [13] for various generalizations). The BRST construction for such algebras was discussed in [14], [15]. Recently, it was shown that a special class of nonlinear gauge algebras arises in higher spin theory on AdS space [16] where the corresponding BRST charge has also been found.

From a general point of view the nonlinear algebras considered in [12], [13], [16] are characterized by the following property. All of them can be described in terms of constraints T_{α} satisfying the relation (1) with nonconstant structure functions which form a finite order polynomial in the constraints T_{α}

$$f^{\gamma}_{\alpha\beta} = F_{\alpha\beta}^{\ \ \gamma} + V^{(1)\gamma\beta_1}_{\alpha\beta}T_{\beta_1} + \cdots + V^{(n-1)\gamma\beta_1\ldots\beta_{n-1}}_{\alpha\beta}T_{\beta_1}\cdots T_{\beta_{n-1}} \tag{4}$$

where $F_{\alpha\beta}^{\ \ \gamma}, V^{(1)\gamma\beta_1}_{\alpha\beta}, \ldots, V^{(n-1)\gamma\beta_1\ldots\beta_{n-1}}_{\alpha\beta}$ are constants. Construction of the nilpotent BRST charge for quadratically nonlinear algebras ($V^{(2)\gamma\beta_1\beta_2}_{\alpha\beta} = \ldots = V^{(n-1)\gamma\beta_1\ldots\beta_{n-1}}_{\alpha\beta} = 0$) subjected to a special assumption concerning structure constants $V^{(1)\gamma\delta}_{\alpha\beta} = V^{\gamma\delta}_{\alpha\beta}$ (see below) was performed in [15] with the result

$$\mathcal{Q} = c^{\alpha}T_{\alpha} - \frac{1}{2}c^{\alpha}c^{\beta}F^{\gamma}_{\alpha\beta}\mathcal{P}_{\gamma} - \frac{1}{2}c^{\alpha}c^{\beta}V^{\gamma}_{\alpha\beta}\mathcal{P}_{\gamma} - \frac{1}{24}c^{\alpha}c^{\beta}c^{\gamma}c^{\delta}V^{\mu\nu}_{\alpha\beta}V^{\rho\sigma}_{\gamma\delta}F^{\lambda}_{\mu\rho}\mathcal{P}_{\nu}\mathcal{P}_{\sigma}\mathcal{P}_{\lambda}. \tag{5}$$

As to general nonlinear algebras of the form (4), to our knowledge, the problem of construction of a nilpotent the BRST charge is open in this case.

In the present paper, we investigate construction of the nilpotent BRST charge for nonlinear algebras (4) and find some special restrictions on structure constants when a nilpotent BRST charge can be presented in the simplest form.

The paper is organized as follows. In Section 2 we construct the classical nilpotent BRST charge for quadratically nonlinear algebras of constraints. In Section 3 we extend our consideration of a classical nilpotent BRST charge for arbitrary nonlinear algebras and find the unique form of the BRST charge. In Section 4 concluding remarks are given.

2 BRST charge for quadratically nonlinear algebras

Let us consider a set of constraints $T_{\alpha} = T_{\alpha}(p,q)$ which satisfy the following (quadratically nonlinear) algebra

$$\{T_{\alpha}, T_{\beta}\} = F_{\alpha\beta}^{\ \ \gamma}T_{\gamma} + V^{\gamma\delta}_{\alpha\beta}T_{\delta}T_{\gamma} \tag{6}$$

where structure constants $F_{\alpha\beta}^{\ \ \gamma}$ and $V^{\gamma\delta}_{\alpha\beta}$ obey the symmetry properties

$$F_{\alpha\beta}^{\ \ \gamma} = -F_{\beta\alpha}^{\ \ \gamma}, \quad V^{\gamma\delta}_{\alpha\beta} = -V^{\gamma\delta}_{\beta\alpha} = V^{\delta\gamma}_{\alpha\beta}. \tag{7}$$

The Jacobi identities for (6) read

$$F_{[\alpha\beta}{}^{\gamma}F_{\lambda]\gamma}{}^{\delta}=0, \tag{8}$$

$$F_{[\alpha\beta}{}^{\rho}V^{\mu\nu}_{\lambda]\rho}+V^{\rho(\mu}_{[\alpha\beta}F_{\lambda]\rho}{}^{\nu)}=0, \tag{9}$$

$$V^{\rho(\mu}_{[\alpha\beta}V^{\nu\gamma)}_{\lambda]\rho}=0, \tag{10}$$

where symbols () and [ ] denote symmetrization and antisymmetrization with respect to indices included in these brackets respectively.

Construction of the classical BRST charge [4],[5] for a given set of constraints (6) involves introducing for each constraint T_α an anticommuting ghost c^α and an anticommuting momentum $\mathcal{P}_\alpha$ having the following distribution of the Grassmann parity $\varepsilon(c^\alpha)=\varepsilon(\mathcal{P}_\alpha)=1$ and the ghost number $gh(c^\alpha)=-gh(\mathcal{P}_\alpha)=1$ and obeying the relations

$$\{c^\alpha,\mathcal{P}_\beta\}=\delta^\alpha_\beta,\quad \{c^\alpha,c^\beta\}=0,\quad \{\mathcal{P}_\alpha,\mathcal{P}_\beta\}=0,\quad \{c^\alpha,T_\beta\}=0,\quad \{\mathcal{P}_\alpha,T_\beta\}=0. \tag{11}$$

The classical BRST charge $\mathcal{Q}$ is defined as a solution to the equation

$$\{\mathcal{Q},\mathcal{Q}\}=0 \tag{12}$$

being odd function of variables $(p,q,c,\mathcal{P})$ with the ghost number $gh(\mathcal{Q})=1$ and satisfying the boundary condition

$$\left.\frac{\partial\mathcal{Q}}{\partial c^\alpha}\right|_{c=0}=T_\alpha. \tag{13}$$

We look for a solution to this problem in the form of the power-series expansions in the ghost variables

$$\mathcal{Q}=c^\alpha T_\alpha+\sum_{k=1}^{\infty}c^{\alpha_1}c^{\alpha_2}\cdots c^{\alpha_{k+1}}U^{(k)\beta_1\beta_2\ldots\beta_k}_{\alpha_1\alpha_2\ldots\alpha_{k+1}}\mathcal{P}_{\beta_1}\mathcal{P}_{\beta_2}\cdots\mathcal{P}_{\beta_k}=c^\alpha T_\alpha+\sum_{k=1}^{\infty}\mathcal{Q}_{k+1}, \tag{14}$$

where the structure functions $U^{(k)}$ are totally antisymmetric in both upper and lower indices.

In lower order we have the following requirement for $\mathcal{Q}$ to be nilpotent

$$c^{\alpha_1}c^{\alpha_2}\Big(\{T_{\alpha_1},T_{\alpha_2}\}+2U^{(1)\beta_1}_{\alpha_1\alpha_2}T_{\beta_1}\Big)=0\quad\Rightarrow\quad\{T_{\alpha_1},T_{\alpha_2}\}+2U^{(1)\beta_1}_{\alpha_1\alpha_2}T_{\beta_1}=0. \tag{15}$$

It leads to the structure function $U^{(1)}$ in the form

$$U^{(1)\gamma}_{\alpha\beta}=-\frac{1}{2}(F_{\alpha\beta}{}^{\gamma}+V^{\gamma\delta}_{\alpha\beta}T_\delta) \tag{16}$$

and the contribution of the second order in ghosts c^α for $\mathcal{Q}$

$$\mathcal{Q}_2=-\frac{1}{2}c^\alpha c^\beta(F_{\alpha\beta}{}^{\gamma}+V^{\gamma\delta}_{\alpha\beta}T_\delta)\mathcal{P}_\gamma. \tag{17}$$

Condition of nilpotency for $\mathcal{Q}$ in the next order has the form

$$c^{\alpha_1}c^{\alpha_2}c^{\alpha_3}\Big(\{U^{(1)\beta_1}_{\alpha_1\alpha_2},T_{\alpha_3}\}+2U^{(1)\gamma}_{\alpha_1\alpha_2}U^{(1)\beta_1}_{\alpha_3\gamma}+2U^{(2)\beta_2\beta_1}_{\alpha_1\alpha_2\alpha_3}T_{\beta_2}\Big)\mathcal{P}_{\beta_1}=0. \tag{18}$$

Using (6) and Jacobi identities (8)-(10) one can rewrite the equation (18) as

$$c^{\alpha_1}c^{\alpha_2}c^{\alpha_3}\Big(-V^{\gamma\rho}_{\alpha_1\alpha_2}V^{\sigma\beta_1}_{\alpha_3\gamma}T_\rho T_\sigma+4U^{(2)\beta_2\beta_1}_{\alpha_1\alpha_2\alpha_3}T_{\beta_2}\Big)\mathcal{P}_{\beta_1}=0 \tag{19}$$

or in the form

$$c^{\alpha_1}c^{\alpha_2}c^{\alpha_3}\Big(-V^{\gamma\rho}_{[\alpha_1\alpha_2}V^{\sigma\beta_1}_{\alpha_3]\gamma}T_\rho T_\sigma+12U^{(2)\beta_2\beta_1}_{\alpha_1\alpha_2\alpha_3}T_{\beta_2}\Big)\mathcal{P}_{\beta_1}=0 \tag{20}$$

Let us introduce the quantity $K^{\alpha}_{\alpha_1\alpha_2\alpha_3}$

$$K^{\alpha}_{\alpha_1\alpha_2\alpha_3} = V^{\gamma\rho}_{[\alpha_1\alpha_2}V^{\sigma\alpha}_{\alpha_3]\gamma}T_\rho T_\sigma. \tag{21}$$

From the Jacobi identities (10) it follows that

$$K^{\alpha}_{\alpha_1\alpha_2\alpha_3}T_\alpha = 0. \tag{22}$$

This means that $K^{\alpha}_{\alpha_1\alpha_2\alpha_3}$ can be presented in the form

$$K^{\alpha}_{\alpha_1\alpha_2\alpha_3} = K^{[\alpha\beta]}_{\alpha_1\alpha_2\alpha_3}T_\beta, \quad K^{[\alpha\beta]}_{\alpha_1\alpha_2\alpha_3} = -K^{[\beta\alpha]}_{\alpha_1\alpha_2\alpha_3}. \tag{23}$$

In its turn, $K^{[\alpha\beta]}_{\alpha_1\alpha_2\alpha_3}$ depends on T_α linearly (see (21) and (23))

$$K^{[\alpha\beta]}_{\alpha_1\alpha_2\alpha_3} = K^{[\alpha\beta]\sigma}_{\alpha_1\alpha_2\alpha_3}T_\sigma. \tag{24}$$

In terms of these quantities the structure functions $U^{(2)}$ read

$$U^{(2)\beta_1\beta_2}_{\alpha_1\alpha_2\alpha_3} = -\frac{1}{12}K^{[\beta_1\beta_2]\sigma}_{\alpha_1\alpha_2\alpha_3}T_\sigma \tag{25}$$

Taking into account (21), (23), (38) and the Jacobi identities (10) we obtain the following equations to define the explicit form of $K^{[\alpha\beta]\sigma}_{\alpha_1\alpha_2\alpha_3}$

$$K^{[\alpha\beta]\sigma}_{\alpha_1\alpha_2\alpha_3} + K^{[\alpha\sigma]\beta}_{\alpha_1\alpha_2\alpha_3} = -V^{\gamma\alpha}_{[\alpha_1\alpha_2}V^{\beta\sigma}_{\alpha_3]\gamma}. \tag{26}$$

This is a basic equation for determining the quantity $U^(2)$ (25).

To solve the equations (26) we note that if it has a solution $K^{[\alpha\beta]\sigma}_{\alpha_1\alpha_2\alpha_3}$ then

$$K^{[\alpha\beta]\sigma}_{\alpha_1\alpha_2\alpha_3} + X^{[\alpha\beta\sigma]}_{[\alpha_1\alpha_2\alpha_3]} \tag{27}$$

will be a solution to (26) as well. In (27) $X^{[\alpha\beta\sigma]}_{[\alpha_1\alpha_2\alpha_3]}$ is totally antisymmetric in both upper and lower indices. It is known [6] that this arbitrariness can be removed by a canonical transformation of the BRST charge which does not change the boundary condition (13). From (21) and (26) one can try to find $K^{[\alpha\beta]\sigma}_{\alpha_1\alpha_2\alpha_3}$ as a linear combination of structures $V^{\gamma\alpha}_{[\alpha_1\alpha_2}V^{\beta\sigma}_{\alpha_3]\gamma}$ (in Appendix A we explicitly verify this proposal as well as the arbitrariness in solutions for the simplest case of three constraints in the algebra (6)). Therefore we can propose

$$K^{[\alpha\beta]\sigma}_{\alpha_1\alpha_2\alpha_3} = C^{[\alpha\beta]\sigma}_{\lambda(\mu\nu)}V^{\gamma\lambda}_{[\alpha_1\alpha_2}V^{\mu\nu}_{\alpha_3]\gamma} \tag{28}$$

where $C^{[\alpha\beta]\sigma}_{\lambda(\mu\nu)}$ is a matrix constructed from the unit matrices δ^α_μ. It is not difficult to find a general structure of $C^{[\alpha\beta]\sigma}_{\lambda(\mu\nu)}$ with the required symmetry properties

$$C^{[\alpha\beta]\sigma}_{\lambda(\mu\nu)} = C\Big(\delta^\alpha_\lambda\delta^\beta_\mu\delta^\sigma_\nu + \delta^\alpha_\lambda\delta^\sigma_\mu\delta^\beta_\nu - \delta^\alpha_\mu\delta^\beta_\lambda\delta^\sigma_\nu - \delta^\alpha_\nu\delta^\beta_\lambda\delta^\sigma_\mu\Big) \tag{29}$$

where C is a constant. Returning with this result in (26) and using the Jacobi identities (10) we obtain

$$(6C+1)V^{\gamma\lambda}_{[\alpha_1\alpha_2}V^{\mu\nu}_{\alpha_3]\gamma} = 0. \tag{30}$$

Therefore to this order we have two solutions to the nilpotency condition. The first one corresponds to

$$C = -\frac{1}{6} \tag{31}$$

and leads to

$$K^{[\alpha\beta]\sigma}_{\alpha_1\alpha_2\alpha_3} = -\frac{1}{3}\Big(V^{\gamma\alpha}_{[\alpha_1\alpha_2}V^{\beta\sigma}_{\alpha_3]\gamma} - V^{\gamma\beta}_{[\alpha_1\alpha_2}V^{\alpha\sigma}_{\alpha_3]\gamma}\Big) = -\frac{1}{3}V^{\gamma[\alpha}_{[\alpha_1\alpha_2}V^{\beta]\sigma}_{\alpha_3]\gamma}. \tag{32}$$

Therefore

$$U^{(2)\beta_1\beta_2}_{\alpha_1\alpha_2\alpha_3} = \frac{1}{36}V^{\gamma[\beta_1}_{[\alpha_1\alpha_2}V^{\beta_2]\sigma}_{\alpha_3]\gamma}T_\sigma \tag{33}$$

and

$$\mathcal{Q}_3 = \frac{1}{6}c^{\alpha_1}c^{\alpha_2}c^{\alpha_3}V^{\gamma\beta_1}_{\alpha_1\alpha_2}V^{\beta_2\sigma}_{\alpha_3\gamma}T_\sigma\mathcal{P}_{\beta_1}\mathcal{P}_{\beta_2} \tag{34}$$

The second possibility corresponds to restriction on structure functions of nonlinear algebras

$$V^{\gamma\lambda}_{[\alpha_1\alpha_2}V^{\mu\nu}_{\alpha_3]\gamma} = 0. \tag{35}$$

This means that $K^{[\alpha\beta]\sigma}_{\alpha_1\alpha_2\alpha_3} = 0$ and

$$U^{(2)\beta_1\beta_2}_{\alpha_1\alpha_2\alpha_3} = 0, \quad \mathcal{Q}_3 = 0. \tag{36}$$

In [15] a class of quadratically nonlinear algebras subjected to restrictions

$$V^{\gamma\lambda}_{\alpha_1\alpha_2}V^{\mu\nu}_{\alpha_3\gamma} = 0. \tag{37}$$

was investigated to construct a nilpotent BRST charge. These restrictions look more stronger in comparison with (35). They are not the direct consequence of (35) and look like additional restrictions which are not dictated by solutions to the nilpotency conditions in the third order. We point out that the Jacobi identities (10) are satisfied in the case of conditions (35). In what follows we suppose fulfilment of conditions (35).

Using these results we rewrite the condition of nilpotency in the forth order in the ghost fields c as follows

$$c^{\alpha_1}c^{\alpha_2}c^{\alpha_3}c^{\alpha_4}\Big(\{U^{(1)\beta_1}_{\alpha_1\alpha_2}, U^{(1)\beta_2}_{\alpha_3\alpha_4}\} + 6U^{(3)\beta_1\beta_2\beta_3}_{\alpha_1\alpha_2\alpha_3\alpha_4}T_{\beta_3}\Big)\mathcal{P}_{\beta_1}\mathcal{P}_{\beta_2} = 0, \tag{38}$$

or

$$c^{\alpha_1}c^{\alpha_2}c^{\alpha_3}c^{\alpha_4}\Big(V^{\beta\beta_1}_{\alpha_1\alpha_2}V^{\beta_2\gamma}_{\alpha_3\alpha_4}F_{\beta\gamma}^{\ \ \beta_3} + V^{\beta\beta_1}_{\alpha_1\alpha_2}V^{\beta_2\gamma}_{\alpha_3\alpha_4}V^{\beta_3\sigma}_{\beta\gamma}T_\sigma + 24U^{(3)\beta_1\beta_2\beta_3}_{\alpha_1\alpha_2\alpha_3\alpha_4})T_{\beta_3}\mathcal{P}_{\beta_1}\mathcal{P}_{\beta_2} = 0. \tag{39}$$

To define the structure function $U^{(3)}$ correctly we have to antisymmetrize quantities presented in (39) in indices $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and $\beta_1, \beta_2, \beta_3$. Taking into account that expressions in (38) are antisymmetrical in β_1, β_2 the antisymmetrization in $\beta_1, \beta_2, \beta_3$ is trivial

$$[123] = [[12]3]] = [12]3 + [23]1 + [31]2 = [12]3 + cycle(123).$$

The antisymmetrization in $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ is performed for two sets of antisymmetric indices $[\alpha_1\alpha_2]$ and $[\alpha_3\alpha_4]$. Symbolically, we can write this procedure as follows

$$[1234] = [[12][34]] = [12][34] + [23][14] + [31][24] + [24][31] + [34][12] + [14][23].$$

Then from (39) we obtain the equations for structure functions $U^{(3)}$

$$V^{\beta\beta_1}_{[\alpha_1\alpha_2}V^{\beta_2\gamma}_{\alpha_3\alpha_4]}F_{\beta\gamma}^{\ \ \beta_3}T_{\beta_3} + V^{\beta\beta_1}_{[\alpha_1\alpha_2}V^{\beta_2\gamma}_{\alpha_3\alpha_4]}V^{\beta_3\sigma}_{\beta\gamma}T_\sigma T_{\beta_3} + 144U^{(3)\beta_1\beta_2\beta_3}_{\alpha_1\alpha_2\alpha_3\alpha_4}T_{\beta_3} = 0, \tag{40}$$

where the following symmetry properties

$$V^{\beta\beta_1}_{[\alpha_1\alpha_2}V^{\beta_2\gamma}_{\alpha_3\alpha_4]}F_{\beta\gamma}^{\ \ \beta_3} = -V^{\beta\beta_2}_{[\alpha_1\alpha_2}V^{\beta_1\gamma}_{\alpha_3\alpha_4]}F_{\beta\gamma}^{\ \ \beta_3}, \quad V^{\beta\beta_1}_{[\alpha_1\alpha_2}V^{\beta_2\gamma}_{\alpha_3\alpha_4]}V^{\beta_3\sigma}_{\beta\gamma} = -V^{\beta\beta_2}_{[\alpha_1\alpha_2}V^{\beta_1\gamma}_{\alpha_3\alpha_4]}V^{\beta_3\sigma}_{\beta\gamma}, \tag{41}$$

were used. Moreover, the following conditions of consistency are satisfied

$$V^{\beta\beta_1}_{[\alpha_1\alpha_2}V^{\gamma(\beta_2}_{\alpha_3\alpha_4]}F_{\beta\gamma}^{\ \ \beta_3)} = 0, \quad V^{\beta\beta_1}_{[\alpha_1\alpha_2}V^{\gamma(\beta_2}_{\alpha_3\alpha_4]}V^{\beta_3\sigma)}_{\beta\gamma} = 0 \tag{42}$$

which are consequence of the Jacobi identities and consistent with (35) (see Appendix B). From (41) and (42) it follows that $V^{\beta\beta_1}_{[\alpha_1\alpha_2}V^{\beta_2\gamma}_{\alpha_3\alpha_4]}F_{\beta\gamma}^{\ \ \beta_3}$ are totally antisymmetric in indices $\beta_1, \beta_2, \beta_3$, and repeat the symmetry properties of $U^{(3)\beta_1\beta_2\beta_3}_{\alpha_1\alpha_2\alpha_3\alpha_4}$. Moreover, when restrictions (35) are satisfied then one can prove (see Appendix B) that

$$V^{\beta\beta_1}_{[\alpha_1\alpha_2}V^{\gamma\beta_2}_{\alpha_3\alpha_4]}V^{\beta_3\sigma}_{\beta\gamma} = 0. \tag{43}$$

Then, finally we get the solution for the structure functions $U^{(3)}$

$$U^{(3)\beta_1\beta_2\beta_3}_{\alpha_1\alpha_2\alpha_3\alpha_4} = -\frac{1}{144}V^{\beta\beta_1}_{[\alpha_1\alpha_2}V^{\beta_2\gamma}_{\alpha_3\alpha_4]}F_{\beta\gamma}^{\ \ \beta_3} \tag{44}$$

and for the BRST charge in the forth order

$$\mathcal{Q}_4 = -\frac{1}{24}c^{\alpha_1}c^{\alpha_2}c^{\alpha_3}c^{\alpha_4}V^{\beta\beta_1}_{\alpha_1\alpha_2}V^{\beta_2\gamma}_{\alpha_3\alpha_4}F_{\beta\gamma}^{\ \ \beta_3}\mathcal{P}_{\beta_1}\mathcal{P}_{\beta_2}\mathcal{P}_{\beta_3}. \tag{45}$$

Therefore for any theory with quadratically nonlinear algebra (6) subjected to the Jacobi identities (8) and (9) as well as the additional restrictions (35) and

$$V^{\beta\beta_1}_{[\alpha_1\alpha_2}V^{\beta_2\gamma}_{\alpha_3\alpha_4]}F_{\beta\gamma}^{\ \ \beta_3} = 0 \tag{46}$$

there exists a unique form of nilpotent the BRST charge

$$\mathcal{Q} = c^\alpha T_\alpha - \frac{1}{2}c^\alpha c^\beta (F_{\alpha\beta}^{\ \ \gamma} + V^{\gamma\delta}_{\alpha\beta}T_\delta)\mathcal{P}_\gamma. \tag{47}$$

3 BRST charge for generic nonlinear algebras

Nonlinear algebras are defined by the following relations

$$\{T_\alpha, T_\beta\} = F_{\alpha\beta}^{\ \ \gamma}T_\gamma + V^{(1)\alpha_1\alpha_2}_{\alpha\beta}T_{\alpha_1}T_{\alpha_2} + V^{(2)\alpha_1\alpha_2\alpha_3}_{\alpha\beta}T_{\alpha_1}T_{\alpha_2}T_{\alpha_3} + \cdots + V^{(n-1)\alpha_1\ldots\alpha_n}_{\alpha\beta}T_{\alpha_1}\cdots T_{\alpha_n}, \tag{48}$$

where structure constants $F_{\alpha\beta}^{\ \ \gamma}$ and $V^{(k-1)\alpha_1\ldots\alpha_k}_{\alpha\beta}$ $(k = 2, 3, ..., n)$ are antisymmetric in lower indices and $V^{(k-1)\alpha_1\ldots\alpha_k}_{\alpha\beta}$ $(k = 2, 3, ..., n)$ are totally symmetric in upper indices.

The Jacobi identities for (48) have the form

$$F_{[\alpha\beta}^{\ \ \gamma}F_{\lambda]\gamma}^{\ \ \delta} = 0\ , \tag{49}$$

$$F_{[\alpha\beta}^{\ \ \rho}V^{(1)\beta_1\beta_2}_{\lambda]\rho} + V^{(1)\rho(\beta_1}_{[\alpha\beta}F_{\lambda]\rho}^{\ \ \beta_2)} = 0\ , \tag{50}$$

$$F_{[\alpha\beta}^{\ \ \rho}V^{(m)\beta_1\ldots\beta_m\beta_{m+1}}_{\lambda]\rho} + V^{(m)\rho(\beta_1\ldots\beta_m}_{[\alpha\beta}F_{\lambda]\rho}^{\ \ \beta_{m+1})} + \\ + \sum_{k=1}^{m-1}\frac{(k+1)!(m-k+1)!}{(m+1)!}V^{(k)\rho(\beta_1\ldots\beta_k}_{[\alpha\beta}V^{(m-k)\beta_{k+1}\ldots\beta_{m+1})}_{\lambda]\rho} = 0\ (m = 2, 3, ..., n-1)\ , \tag{51}$$

$$\sum_{k=m-n+1}^{n-1}\frac{(k+1)!(m-k+1)!}{(m+1)!}V^{(k)\rho(\beta_1\ldots\beta_k}_{[\alpha\beta}V^{(m-k)\beta_{k+1}\ldots\beta_{m+1})}_{\lambda]\rho} = 0\ (m = n, ..., 2n-2)\ . \tag{52}$$

In Eqs. (51), (52) symmetrization includes two sets of symmetric indices. We assume that in the symmetrization only one representative among equivalent ones obtained by permutation of indices into these sets is presented.

Consider now construction of the classical BRST charge $\mathcal{Q}$ for the algebra (48). Equation for the structure function $U^{(1)}$ has the form

$$c^\alpha c^\beta\Big(\{T_\alpha,T_\beta\}+2U^{(1)\gamma}_{\alpha\beta}T_\gamma\Big)=0 \tag{53}$$

or equivalently

$$c^\alpha c^\beta\Big(F_{\alpha\beta}^{\ \ \gamma}+V^{(1)\gamma\alpha_1}_{\alpha\beta}T_{\alpha_1}+\cdots+V^{(n-1)\gamma\alpha_1\ldots\alpha_{n-1}}_{\alpha\beta}T_{\alpha_1}\cdots T_{\alpha_{n-1}}+2U^{(1)\gamma}_{\ \ \alpha\beta}\Big)T_\gamma=0 \tag{54}$$

with the evident solution

$$U^{(1)\gamma}_{\ \ \alpha\beta}=-\frac{1}{2}\Big(F_{\alpha\beta}^{\ \ \gamma}+V^{(1)\gamma\alpha_1}_{\alpha\beta}T_{\alpha_1}+\cdots+V^{(n-1)\gamma\alpha_1\ldots\alpha_{n-1}}_{\alpha\beta}T_{\alpha_1}\cdots T_{\alpha_{n-1}}\Big). \tag{55}$$

In the next order we have the following equation of nilpotency

$$c^{\alpha_1}c^{\alpha_2}c^{\alpha_3}\Big(\{U^{(1)\beta_1}_{\alpha_1\alpha_2},T_{\alpha_3}\}+2U^{(1)\gamma}_{\alpha_1\alpha_2}U^{(1)\beta_1}_{\alpha_3\gamma}+2U^{(2)\gamma\beta_1}_{\alpha_1\alpha_2\alpha_3}T_\gamma\Big)\mathcal{P}_{\beta_1}=0. \tag{56}$$

To solve this equation for the structure function $U^{(2)}$ let us consider the equality

$$\{T_{\alpha_1},T_{\alpha_2}\}+2U^{(1)\beta_1}_{\alpha_1\alpha_2}T_{\beta_1}=0 \tag{57}$$

and the consequence from it

$$\{\{T_{\alpha_1},T_{\alpha_2}\},T_{\alpha_3}\}+2U^{(1)\beta_1}_{\alpha_1\alpha_2}\{T_{\beta_1},T_{\alpha_3}\}+2\{U^{(1)\beta_1}_{\alpha_1\alpha_2},T_{\alpha_3}\}T_{\beta_1}=0. \tag{58}$$

Using Eq.(57) and the symmetry properties of $U^{(1)}$ we can rewrite the last equation asfollows

$$\{\{T_{\alpha_1},T_{\alpha_2}\},T_{\alpha_3}\}+2\Big(2U^{(1)\gamma}_{\alpha_1\alpha_2}U^{(1)\beta_1}_{\alpha_3\gamma}+\{U^{(1)\beta_1}_{\alpha_1\alpha_2},T_{\alpha_3}\}\Big)T_{\beta_1}=0. \tag{59}$$

Taking into account the Jacobi identities for Poisson bracket (see (49)–(52)) $\{\{T_{\alpha_1},T_{\alpha_2}\},T_{\alpha_3}\}+cycle(\alpha_1\alpha_2\alpha_3)=0$, from (59) it follows that

$$\Big(2U^{(1)\gamma}_{[\alpha_1\alpha_2}U^{(1)\beta_1}_{\alpha_3]\gamma}+\{U^{(1)\beta_1}_{[\alpha_1\alpha_2},T_{\alpha_3]}\}\Big)T_{\beta_1}=0. \tag{60}$$

Let us introduce the quantities

$$K^{\beta_1}_{\alpha_1\alpha_2\alpha_3}=2U^{(1)\gamma}_{[\alpha_1\alpha_2}U^{(1)\beta_1}_{\alpha_3]\gamma}+\{U^{(1)\beta_1}_{[\alpha_1\alpha_2},T_{\alpha_3]}\} \tag{61}$$

obeying the property

$$K^{\beta_1}_{\alpha_1\alpha_2\alpha_3}T_{\beta_1}=0 \tag{62}$$

due to (60). This means that $K^{\beta_1}_{\alpha_1\alpha_2\alpha_3}$ can be presented in the form

$$K^{\beta_1}_{\alpha_1\alpha_2\alpha_3}=K^{[\beta_1\beta_2]}_{\alpha_1\alpha_2\alpha_3}T_{\beta_2},\quad K^{[\beta_1\beta_2]}_{\alpha_1\alpha_2\alpha_3}=-K^{[\beta_2\beta_1]}_{\alpha_1\alpha_2\alpha_3}. \tag{63}$$

In terms of $K^{[\beta_1\beta_2]}_{\alpha_1\alpha_2\alpha_3}$ solution for structure functions $U^{(2)\beta_1\beta_2}_{\alpha_1\alpha_2\alpha_3}$ reads

$$U^{(2)\beta_1\beta_2}_{\alpha_1\alpha_2\alpha_3}=\frac{1}{6}K^{[\beta_1\beta_2]}_{\alpha_1\alpha_2\alpha_3}. \tag{64}$$

Straightforward calculations of $K^{\beta_1}_{\alpha_1\alpha_2\alpha_3}$ lead to the following result

$$\begin{aligned}
K^{\beta_1}_{\alpha_1\alpha_2\alpha_3} &= \frac{1}{2}\Big(F^{\sigma}_{[\alpha_1\alpha_2}F^{\beta_1}_{\alpha_3]\sigma} + \big(F^{\sigma}_{[\alpha_1\alpha_2}V^{(1)\beta_1\sigma_1}_{\alpha_3]\sigma} + V^{(1)\sigma(\sigma_1}_{[\alpha_1\alpha_2}F^{\beta_1)}_{\alpha_3]\sigma}\big)T_{\sigma_1} + \\
&+ \sum_{m=2}^{n-1}\big(F^{\sigma}_{[\alpha_1\alpha_2}V^{(m)\beta_1\sigma_1\ldots\sigma_m}_{\alpha_3]\sigma} + V^{(m)\sigma(\sigma_1\ldots\sigma_m}_{[\alpha_1\alpha_2}F^{\beta_1)}_{\alpha_3]\sigma}\big)T_{\sigma_1}\cdots T_{\sigma_m} + \\
&+ \sum_{m=2}^{n-1}\sum_{k=1}^{m-1}\frac{k!(m-k)!}{m!}V^{(k)\sigma(\beta_1\sigma_1\ldots\sigma_{k-1}}_{[\alpha_1\alpha_2}V^{(m-k)\sigma_k\ldots\sigma_m)}_{\alpha_3]\sigma}\ T_{\sigma_1}\cdots T_{\sigma_m} + \\
&+ \sum_{m=2}^{n-1}\sum_{k=1}^{m-1}\frac{k!(m-k)!(m-k)}{m!}V^{(k)\sigma\beta_1(\sigma_1\ldots\sigma_{k-1}}_{[\alpha_1\alpha_2}V^{(m-k)\sigma_k\ldots\sigma_m)}_{\alpha_3]\sigma}\ T_{\sigma_1}\cdots T_{\sigma_m} + \\
&+ \sum_{m=n+1}^{2n-2}\ \sum_{k=m-n+1}^{n-1}\frac{k!(m-k)!}{m!}V^{(k)\sigma(\beta_1\sigma_1\ldots\sigma_{k-1}}_{[\alpha_1\alpha_2}V^{(m-k)\sigma_k\ldots\sigma_m)}_{\alpha_3]\sigma}\ T_{\sigma_1}\cdots T_{\sigma_m} + \\
&+ \sum_{m=n+1}^{2n-2}\ \sum_{k=m-n+1}^{n-1}\frac{k!(m-k)!(m-k)}{m!}V^{(k)\sigma\beta_1(\sigma_1\ldots\sigma_{k-1}}_{[\alpha_1\alpha_2}V^{(m-k)\sigma_k\ldots\sigma_m)}_{\alpha_3]\sigma}\ T_{\sigma_1}\cdots T_{\sigma_m}\Big)\,. \qquad (65)
\end{aligned}$$

In deriving (65) the relations

$$\begin{aligned}
&V^{(k)\sigma\beta_1(\sigma_1\ldots\sigma_k}_{[\alpha_1\alpha_2}V^{(m-k)\sigma_{k+1}\ldots\sigma_m)\beta_1}_{\alpha_3]\sigma} + (m-k+1)V^{(k)\sigma\beta_1(\sigma_1\ldots\sigma_{k-1}}_{[\alpha_1\alpha_2}V^{(m-k)\sigma_k\ldots\sigma_m)}_{\alpha_3]\sigma} = \qquad (66)\\
&= V^{(k)\sigma(\beta_1\sigma_1\ldots\sigma_{k-1}}_{[\alpha_1\alpha_2}V^{(m-k)\sigma_k\ldots\sigma_m)}_{\alpha_3]\sigma} + (m-k)V^{(k)\sigma\beta_1(\sigma_1\ldots\sigma_{k-1}}_{[\alpha_1\alpha_2}V^{(m-k)\sigma_k\ldots\sigma_m)}_{\alpha_3]\sigma}
\end{aligned}$$

were used. Taking into account the Jacobi identities (49)– (51) we can rewrite (65) in the form

$$\begin{aligned}
K^{\beta_1}_{\alpha_1\alpha_2\alpha_3} &= \frac{1}{2}\Big(-\sum_{m=2}^{n-1}\sum_{k=1}^{m-1}\frac{k!k\ (m-k)!(m-k)}{(m+1)!}V^{(k)\sigma(\beta_1\sigma_1\ldots\sigma_{k-1}}_{[\alpha_1\alpha_2}V^{(m-k)\sigma_k\ldots\sigma_m)}_{\alpha_3]\sigma}\ T_{\sigma_1}\cdots T_{\sigma_m} + \\
&+ \sum_{m=2}^{n-1}\sum_{k=1}^{m-1}\frac{k!(m-k)!(m-k)}{m!}V^{(k)\sigma\beta_1(\sigma_1\ldots\sigma_{k-1}}_{[\alpha_1\alpha_2}V^{(m-k)\sigma_k\ldots\sigma_m)}_{\alpha_3]\sigma}\ T_{\sigma_1}\cdots T_{\sigma_m} + \\
&+ \sum_{m=n+1}^{2n-2}\ \sum_{k=m-n+1}^{n-1}\frac{k!(m-k)!}{m!}V^{(k)\sigma(\beta_1\sigma_1\ldots\sigma_{k-1}}_{[\alpha_1\alpha_2}V^{(m-k)\sigma_k\ldots\sigma_m)}_{\alpha_3]\sigma}\ T_{\sigma_1}\cdots T_{\sigma_m} + \\
&+ \sum_{m=n+1}^{2n-2}\ \sum_{k=m-n+1}^{n-1}\frac{k!(m-k)!(m-k)}{m!}V^{(k)\sigma\beta_1(\sigma_1\ldots\sigma_{k-1}}_{[\alpha_1\alpha_2}V^{(m-k)\sigma_k\ldots\sigma_m)}_{\alpha_3]\sigma}\ T_{\sigma_1}\cdots T_{\sigma_m}\Big)\,. \qquad (67)
\end{aligned}$$

Now let us require the following additional restrictions on structure constants of the algebra (48)

$$\begin{aligned}
&V^{(k)\sigma\beta_1\sigma_1\ldots\sigma_{k-1}}_{[\alpha_1\alpha_2}V^{(m-k)\sigma_k\ldots\sigma_m}_{\alpha_3]\sigma} = 0, \qquad (68)\\
&k = 1,2,...,n-1,\quad m>k,\quad m = 2,3,...,2n-2.
\end{aligned}$$

Then we obtain

$$K^{\beta_1}_{\alpha_1\alpha_2\alpha_3} = 0, \quad U^{(2)\beta_1\beta_2}_{\alpha_1\alpha_2\alpha_3} = 0, \quad \mathcal{Q}_3 = 0. \qquad (69)$$

Conditions (68) generalize (35) for nonlinear algebras.

With this result we have the following condition of nilpotency in the next order

$$\begin{aligned}
c^{\alpha_1}c^{\alpha_2}c^{\alpha_3}c^{\alpha_4}\qquad &\Big(\bar{V}^{\sigma\beta_1}_{\alpha_1\alpha_2}\bar{V}^{\beta_2\rho}_{\alpha_3\alpha_4}F^{\beta_3}_{\sigma\rho} + \bar{V}^{\sigma\beta_1}_{\alpha_1\alpha_2}\bar{V}^{\beta_2\rho}_{\alpha_3\alpha_4}\bar{V}^{\beta_3\lambda}_{\sigma\rho}T_\lambda + \bar{V}^{\sigma[\beta_1}_{\alpha_1\alpha_2}\tilde{V}^{\beta_2]\rho}_{\alpha_3\alpha_4}F^{\beta_3}_{\sigma\rho} + \bar{V}^{\sigma[\beta_1}_{\alpha_1\alpha_2}\tilde{V}^{\beta_2]\rho}_{\alpha_3\alpha_4}\bar{V}^{\beta_3\lambda}_{\sigma\rho}T_\lambda + \\
&+\tilde{V}^{\sigma\beta_1}_{\alpha_1\alpha_2}\tilde{V}^{\beta_2\rho}_{\alpha_3\alpha_4}F^{\beta_3}_{\sigma\rho} + \tilde{V}^{\sigma\beta_1}_{\alpha_1\alpha_2}\tilde{V}^{\beta_2\rho}_{\alpha_3\alpha_4}\bar{V}^{\beta_3\lambda}_{\sigma\rho}T_\lambda + 24U^{(3)\beta_1\beta_2\beta_3}_{\alpha_1\alpha_2\alpha_3\alpha_4}\Big)T_{\beta_3}\mathcal{P}_{\beta_1}\mathcal{P}_{\beta_2} = 0 \qquad (70)
\end{aligned}$$

where the notations

$$\bar{V}^{\alpha\beta}_{\mu\nu} = \sum_{k=1}^{n-1} V^{(k)\alpha\beta\sigma_1\ldots\sigma_{k-1}}_{\mu\nu} T_{\sigma_1}\cdots T_{\sigma_{k-1}} \tag{71}$$

$$\tilde{V}^{\alpha\beta}_{\mu\nu} = \sum_{k=1}^{n-1} (k-1) V^{(k)\alpha\beta\sigma_1\ldots\sigma_{k-1}}_{\mu\nu} T_{\sigma_1}\cdots T_{\sigma_{k-1}}. \tag{72}$$

were used. From (70) we obtain the following equations to find $U^{(3)\beta_1\beta_2\beta_3}_{\alpha_1\alpha_2\alpha_3\alpha_4}$

$$\bar{V}^{\sigma\beta_1}_{[\alpha_1\alpha_2}\bar{V}^{\beta_2\rho}_{\alpha_3\alpha_4]}F^{\beta_3}_{\sigma\rho}T_{\beta_3} + \bar{V}^{\sigma\beta_1}_{[\alpha_1\alpha_2}\bar{V}^{\beta_2\rho}_{\alpha_3\alpha_4]}\bar{V}^{\beta_3\lambda}_{\sigma\rho}T_\lambda T_{\beta_3} + \bar{V}^{\sigma[\beta_1}_{[\alpha_1\alpha_2}\tilde{V}^{\beta_2]\rho}_{\alpha_3\alpha_4]}F^{\beta_3}_{\sigma\rho}T_{\beta_3} + \bar{V}^{\sigma[\beta_1}_{[\alpha_1\alpha_2}\tilde{V}^{\beta_2]\rho}_{\alpha_3\alpha_4]}\bar{V}^{\beta_3\lambda}_{\sigma\rho}T_\lambda T_{\beta_3} + \\ + \tilde{V}^{\sigma\beta_1}_{[\alpha_1\alpha_2}\tilde{V}^{\beta_2\rho}_{\alpha_3\alpha_4]}F^{\beta_3}_{\sigma\rho}T_{\beta_3} + \tilde{V}^{\sigma\beta_1}_{[\alpha_1\alpha_2}\tilde{V}^{\beta_2\rho}_{\alpha_3\alpha_4]}\bar{V}^{\beta_3\lambda}_{\sigma\rho}T_\lambda T_{\beta_3} + 144U^{(3)\beta_1\beta_2\beta_3}_{\alpha_1\alpha_2\alpha_3\alpha_4}T_{\beta_3} = 0. \tag{73}$$

The conditions of consistency have the form

$$\bar{V}^{\sigma\beta_1}_{[\alpha_1\alpha_2}\bar{V}^{\rho(\beta_2}_{\alpha_3\alpha_4]}F^{\beta_3)}_{\sigma\rho} = 0, \quad \tilde{V}^{\sigma\beta_1}_{[\alpha_1\alpha_2}\tilde{V}^{\rho(\beta_2}_{\alpha_3\alpha_4]}F^{\beta_3)}_{\sigma\rho} = 0, \quad \bar{V}^{\sigma[\beta_1}_{[\alpha_1\alpha_2}\tilde{V}^{\beta_2]\rho}_{\alpha_3\alpha_4]}F^{\beta_3}_{\sigma\rho} + \bar{V}^{\sigma[\beta_1}_{[\alpha_1\alpha_2}\tilde{V}^{\beta_3]\rho}_{\alpha_3\alpha_4]}F^{\beta_2}_{\sigma\rho} = 0, \tag{74}$$

$$\bar{V}^{\sigma\beta_1}_{[\alpha_1\alpha_2}\bar{V}^{\rho(\beta_2}_{\alpha_3\alpha_4]}\bar{V}^{\beta_3\lambda)}_{\sigma\rho} = 0, \quad \tilde{V}^{\sigma\beta_1}_{[\alpha_1\alpha_2}\tilde{V}^{\rho(\beta_2}_{\alpha_3\alpha_4]}\bar{V}^{\beta_3\lambda)}_{\sigma\rho} = 0, \quad \bar{V}^{\sigma[\beta_1}_{[\alpha_1\alpha_2}\tilde{V}^{\beta_2]\rho}_{\alpha_3\alpha_4]}\bar{V}^{\beta_3\lambda}_{\sigma\rho} + cycle(\beta_2,\beta_3,\lambda) = 0. \tag{75}$$

These conditions are satisfied due to the Jacobi identities (49) – (52) and the restrictions (68) (see Appendix B). All terms appearing in the equations (73) are totally antisymmetric in indices $\beta_1, \beta_2, \beta_3$ and repeat the symmetry properties of $U^{(3)\beta_1\beta_2\beta_3}_{\alpha_1\alpha_2\alpha_3\alpha_4}$. We point out that the restrictions (68) lead to equalities

$$\bar{V}^{\sigma\beta_1}_{[\alpha_1\alpha_2}\bar{V}^{\beta_2\rho}_{\alpha_3\alpha_4]}\bar{V}^{\beta_3\lambda}_{\sigma\rho} = 0, \quad \tilde{V}^{\sigma\beta_1}_{[\alpha_1\alpha_2}\tilde{V}^{\beta_2\rho}_{\alpha_3\alpha_4]}\bar{V}^{\beta_3\lambda}_{\sigma\rho} = 0, \quad \bar{V}^{\sigma[\beta_1}_{[\alpha_1\alpha_2}\tilde{V}^{\beta_2]\rho}_{\alpha_3\alpha_4]}\bar{V}^{\beta_3\lambda}_{\sigma\rho} = 0. \tag{76}$$

Therefore we find the solution for structure functions $U^{(3)}$

$$U^{(3)\beta_1\beta_2\beta_3}_{\alpha_1\alpha_2\alpha_3\alpha_4} = -\frac{1}{144}\Big(\bar{V}^{\sigma\beta_1}_{[\alpha_1\alpha_2}\bar{V}^{\beta_2\rho}_{\alpha_3\alpha_4]}F^{\beta_3}_{\sigma\rho} + \bar{V}^{\sigma[\beta_1}_{[\alpha_1\alpha_2}\tilde{V}^{\beta_2]\rho}_{\alpha_3\alpha_4]}F^{\beta_3}_{\sigma\rho} + \tilde{V}^{\sigma\beta_1}_{[\alpha_1\alpha_2}\tilde{V}^{\beta_2\rho}_{\alpha_3\alpha_4]}F^{\beta_3}_{\sigma\rho}\Big) \tag{77}$$

and the BRST charge in the forth order

$$\mathcal{Q}_4 = -\frac{1}{24}c^{\alpha_1}c^{\alpha_2}c^{\alpha_3}c^{\alpha_4}\Big(\bar{V}^{\sigma\beta_1}_{\alpha_1\alpha_2}\bar{V}^{\beta_2\rho}_{\alpha_3\alpha_4}F^{\beta_3}_{\sigma\rho} + 2\bar{V}^{\sigma\beta_1}_{\alpha_1\alpha_2}\tilde{V}^{\beta_2\rho}_{\alpha_3\alpha_4}F^{\beta_3}_{\sigma\rho} + \tilde{V}^{\sigma\beta_1}_{\alpha_1\alpha_2}\tilde{V}^{\beta_2\rho}_{\alpha_3\alpha_4}F^{\beta_3}_{\sigma\rho}\Big)\mathcal{P}_{\beta_1}\mathcal{P}_{\beta_2}\mathcal{P}_{\beta_3}. \tag{78}$$

If we additionally assume the following restrictions on the structure constants

$$\begin{gathered} V^{(k)\beta\beta_1\sigma_1\ldots\sigma_{k-1}}_{[\alpha_1\alpha_2}V^{(m-k)\beta_2\gamma\sigma_k\ldots\sigma_{m-2}}_{\alpha_3\alpha_4]}F^{\beta_3}_{\beta\gamma} = 0, \\ k = 1,...,n-1, \quad m > k, \quad m = 2,...,2n-2, \end{gathered} \tag{79}$$

then we have

$$\bar{V}^{\sigma\beta_1}_{[\alpha_1\alpha_2}\bar{V}^{\beta_2\rho}_{\alpha_3\alpha_4]}F^{\beta_3}_{\sigma\rho} = 0, \quad \bar{V}^{\sigma[\beta_1}_{[\alpha_1\alpha_2}\tilde{V}^{\beta_2]\rho}_{\alpha_3\alpha_4]}F^{\beta_3}_{\sigma\rho} = 0, \quad \tilde{V}^{\sigma\beta_1}_{[\alpha_1\alpha_2}\tilde{V}^{\beta_2\rho}_{\alpha_3\alpha_4]}F^{\beta_3}_{\sigma\rho} = 0 \tag{80}$$

and as the result

$$U^{(3)\beta_1\beta_2\beta_3}_{\alpha_1\alpha_2\alpha_3\alpha_4} = 0, \quad \mathcal{Q}_4 = 0. \tag{81}$$

Therefore as in case of quadratically algebras, for nonlinear algebras (48) there exists a unique form of the nilpotent BRST charge

$$\mathcal{Q} = c^\alpha T_\alpha - \frac{1}{2}c^\alpha c^\beta\Big(F_{\alpha\beta}^{\ \ \gamma} + \bar{V}^{\gamma\beta}_{\alpha\beta}T_\beta\Big)\mathcal{P}_\gamma \tag{82}$$

if conditions (68), (79) are fulfilled.

4 Summary

In this paper we have studied a construction of the nilpotent classical BRST charge for nonlinear algebras of the form (48) which are characterized by the structure constants $F_{\alpha\beta}^{\ \ \gamma}, V_{\alpha\beta}^{(1)\alpha_1\alpha_2}, ..., V_{\alpha\beta}^{(n-1)\alpha_1...\alpha_n}$. For quadratically nonlinear algebras the explicit form of the BRST charge in the third order has been found without any additional restrictions on structure constants. We have constructed the BRST charge up to the forth order in the ghost fields when the structure constants are subject to restrictions (68). We have shown that if the conditions (68) and (79) are satisfied and a set of constraints T_α is linearly independent then the BRST charge is given in the universal form (82). We have proved that suitable quantities in terms of which one can efficiently analyze general nonlinear algebras (48) are $\bar{V}_{\alpha\beta}^{\mu\nu}, \tilde{V}_{\alpha\beta}^{\mu\nu}$.

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Appendix

A Solutions to equations (26) in the simplest case

To support our proposal concerning the structure of solutions to Eqs. (26) as well as the arbitrariness in these solutions, we solve the equations

$$K^{[\alpha\beta]\sigma}_{\alpha_1\alpha_2\alpha_3} + K^{[\alpha\sigma]\beta}_{\alpha_1\alpha_2\alpha_3} = -V^{\gamma\alpha}_{[\alpha_1\alpha_2}V^{\beta\sigma}_{\alpha_3]\gamma}. \tag{A.1}$$

together with the Jacobi identities

$$V^{\gamma\alpha}_{[\alpha_1\alpha_2}V^{\beta\sigma}_{\alpha_3]\gamma} + V^{\gamma\beta}_{[\alpha_1\alpha_2}V^{\sigma\alpha}_{\alpha_3]\gamma} + V^{\gamma\sigma}_{[\alpha_1\alpha_2}V^{\alpha\beta}_{\alpha_3]\gamma} = 0 \tag{A.2}$$

in the simplest case when there are only three constraints ($\alpha = 1, 2, 3$) in the algebra (6). To simplify notations we omit lower indices $[\alpha_1\alpha_2\alpha_3]$ and introduce quantities

$$K^{[\alpha\beta]\sigma} \equiv K^{[\alpha\beta]\sigma}_{\alpha_1\alpha_2\alpha_3}, \quad R^{\alpha(\beta\sigma)} \equiv -V^{\gamma\alpha}_{[\alpha_1\alpha_2}V^{\beta\sigma}_{\alpha_3]\gamma}. \tag{A.3}$$

In terms of $R^{\alpha(\beta\sigma)}$ the Jacobi identities (A.2) are rewritten as

$$R^{\alpha(\beta\sigma)} + R^{\beta(\sigma\alpha)} + R^{\sigma(\alpha\beta)} = 0. \tag{A.4}$$

From (A.1) it follows the complete set of equations

$$\begin{aligned}
&K^{[12]1} = R^{1(12)}, \quad K^{[21]1} = \frac{1}{2}R^{2(11)}, \quad K^{[31]1} = \frac{1}{2}R^{3(11)}, \quad K^{[13]1} = R^{1(13)},\\
&K^{[12]2} = \frac{1}{2}R^{1(22)}, \quad K^{[21]2} = R^{2(12)}, \quad K^{[13]3} = \frac{1}{2}R^{1(33)}, \quad K^{[31]3} = R^{3(13)},\\
&K^{[23]2} = R^{2(23)}, \quad K^{[32]2} = \frac{1}{2}R^{3(22)}, \quad K^{[32]3} = R^{3(23)}, \quad K^{[23]3} = \frac{1}{2}R^{2(33)},\\
&K^{[21]3} + K^{[23]1} = R^{2(13)}, \quad K^{[31]2} + K^{[32]1} = R^{3(12)}, \quad K^{[12]3} + K^{[13]2} = R^{1(23)}.
\end{aligned} \tag{A.5}$$

Consider equations for $K^{[12]1}$ and $K^{[21]1}$. Due to the Jacobi identities (A.4) we have $2R^{1(12)} = -R^{2(11)}$ and therefore the required symmetry property $K^{[12]1} = -K^{[21]1}$ is fulfilled as the consequence of solution to Eqs. (A.5). Moreover we have the following representation of $K^{[12]1}$

$$K^{[12]1} = \frac{1}{3}\Big(R^{1(21)} - R^{2(11)}\Big). \tag{A.6}$$

In a similar way, we obtain

$$\begin{aligned} K^{[13]1} &= \frac{1}{3}\Big(R^{1(31)} - R^{3(11)}\Big), \quad K^{[21]2} = \frac{1}{3}\Big(R^{2(12)} - R^{1(22)}\Big), \\ K^{[31]3} &= \frac{1}{3}\Big(R^{3(13)} - R^{1(33)}\Big), \quad K^{[23]2} = \frac{1}{3}\Big(R^{2(32)} - R^{3(22)}\Big), \\ K^{[32]3} &= \frac{1}{3}\Big(R^{3(23)} - R^{2(33)}\Big). \end{aligned} \tag{A.7}$$

For remaining quantities we propose the following form

$$\begin{aligned} K^{[12]3} &= \frac{1}{3}\Big(R^{1(23)} - R^{2(13)}\Big) + X_1, \quad K^{[21]3} = \frac{1}{3}\Big(R^{2(13)} - R^{1(23)}\Big) + X_2, \\ K^{[13]2} &= \frac{1}{3}\Big(R^{1(32)} - R^{3(12)}\Big) + X_3, \quad K^{[31]2} = \frac{1}{3}\Big(R^{3(12)} - R^{1(32)}\Big) + X_4, \\ K^{[23]1} &= \frac{1}{3}\Big(R^{2(31)} - R^{3(21)}\Big) + X_5, \quad K^{[32]1} = \frac{1}{3}\Big(R^{3(21)} - R^{2(31)}\Big) + X_6. \end{aligned} \tag{A.8}$$

Substituting these quantities into the last line of (A.5) and using the Jacobi identities (A.4), we obtain

$$X_1 + X_3 = 0, \quad X_2 + X_5 = 0, \quad X_4 + X_6 = 0. \tag{A.9}$$

The required symmetry properties of $K^{[12]3}, K^{[13]2}$ and $K^{[23]1}$ lead to relations

$$X_1 = -X_2, \quad X_3 = -X_4, \quad X_5 = -X_6. \tag{A.10}$$

Therefore

$$X_1 = -X_2 = -X_3 = X_4 = X_5 = -X_6 \tag{A.11}$$

and the general solution to Eqs. (A.5) is characterized by the only constant $X_1 \equiv X^{[123]}$ and can be presented in the form

$$K^{[\alpha\beta]\gamma} = \frac{1}{3}\Big(R^{\alpha(\beta\gamma)} - R^{\beta(\alpha\gamma)}\Big) + X^{[\alpha\beta\gamma]}. \tag{A.12}$$

Returning to equations (A.1) we find the general solutions in the form

$$K^{[\alpha\beta]\sigma}_{[\alpha_1\alpha_2\alpha_3]} = -\frac{1}{3}V^{\gamma[\alpha}_{[\alpha_1\alpha_2}V^{\beta]\sigma}_{\alpha_3]\gamma} + X^{[\alpha\beta\sigma]}_{[\alpha_1\alpha_2\alpha_3]} \tag{A.13}$$

proposed in Section 2.

B Proof of identities (42), (74), (75) and (76)

In this Appendix we prove identities used in Sections 2 and 3. Let us start with (42). Consider the Jacobi identities (9) written in the form

$$F^{\gamma}_{\alpha_3\alpha_4}V^{\beta_2\beta_3}_{\beta\gamma} + F^{\gamma}_{\alpha_4\beta}V^{\beta_2\beta_3}_{\alpha_3\gamma} + F^{\gamma}_{\beta\alpha_3}V^{\beta_2\beta_3}_{\alpha_4\gamma} + V^{\gamma(\beta_2}_{\alpha_3\alpha_4}F^{\beta_3)}_{\beta\gamma} + V^{\gamma(\beta_2}_{\alpha_4\beta}F^{\beta_3)}_{\alpha_3\gamma} + V^{\gamma(\beta_2}_{\beta\alpha_3}F^{\beta_3)}_{\alpha_4\gamma} = 0. \tag{B.1}$$

Multiplying these identities by $V^{\beta\beta_1}_{\alpha_1\alpha_2}$ and summarizing over β we have

$$V^{\beta\beta_1}_{\alpha_1\alpha_2}F^{\gamma}_{\alpha_3\alpha_4}V^{\beta_2\beta_3}_{\beta\gamma}+V^{\beta\beta_1}_{\alpha_1\alpha_2}F^{\gamma}_{\alpha_4\beta}V^{\beta_2\beta_3}_{\alpha_3\gamma}+V^{\beta\beta_1}_{\alpha_1\alpha_2}F^{\gamma}_{\beta\alpha_3}V^{\beta_2\beta_3}_{\alpha_4\gamma}+$$
$$V^{\beta\beta_1}_{\alpha_1\alpha_2}V^{\gamma(\beta_2}_{\alpha_3\alpha_4}F^{\beta_3)}_{\beta\gamma}+V^{\beta\beta_1}_{\alpha_1\alpha_2}V^{\gamma(\beta_2}_{\alpha_4\beta}F^{\beta_3)}_{\alpha_3\gamma}+V^{\beta\beta_1}_{\alpha_1\alpha_2}V^{\gamma(\beta_2}_{\beta\alpha_3}F^{\beta_3)}_{\alpha_4\gamma}=0. \tag{B.2}$$

Antisymmetrizing these identities in indices $\alpha_1,\alpha_2,\alpha_3,\alpha_4$ we obtain the relations

$$\begin{aligned}
V^{\beta\beta_1}_{[\alpha_1\alpha_2}F^{\gamma}_{\alpha_3\alpha_4]}V^{\beta_2\beta_3}_{\beta\gamma} \quad &- \quad V^{\beta\beta_1}_{[\alpha_1\alpha_2}F^{\gamma}_{\alpha_3]\beta}V^{\beta_2\beta_3}_{\alpha_4\gamma}+V^{\beta\beta_1}_{[\alpha_1\alpha_2}F^{\gamma}_{\alpha_4]\beta}V^{\beta_2\beta_3}_{\alpha_3\gamma}+\\
&\quad\;\; V^{\beta\beta_1}_{[\alpha_1\alpha_4}F^{\gamma}_{\alpha_3]\beta}V^{\beta_2\beta_3}_{\alpha_2\gamma}+V^{\beta\beta_1}_{[\alpha_2\alpha_3}F^{\gamma}_{\alpha_4]\beta}V^{\beta_2\beta_3}_{\alpha_1\gamma}+\\
V^{\beta\beta_1}_{[\alpha_1\alpha_2}V^{\gamma(\beta_2}_{\alpha_3\alpha_4]}F^{\beta_3)}_{\beta\gamma} \quad &- \quad V^{\beta\beta_1}_{[\alpha_1\alpha_2}V^{\gamma(\beta_2}_{\alpha_3]\beta}F^{\beta_3)}_{\alpha_4\gamma}+V^{\beta\beta_1}_{[\alpha_1\alpha_2}V^{\gamma(\beta_2}_{\alpha_4]\beta}F^{\beta_3)}_{\alpha_3\gamma}+\\
&\quad\;\; V^{\beta\beta_1}_{[\alpha_1\alpha_4}V^{\gamma(\beta_2}_{\alpha_3]\beta}F^{\beta_3)}_{\alpha_2\gamma}+V^{\beta\beta_1}_{[\alpha_2\alpha_3}V^{\gamma(\beta_2}_{\alpha_4]\beta}F^{\beta_3)}_{\alpha_1\gamma}=0.
\end{aligned} \tag{B.3}$$

Note that four last terms are equal to zero because of (35). Using the Jacobi identities (9) we can rewrite (B.3) as follows

$$\begin{aligned}
&V^{\beta\beta_1}_{[\alpha_1\alpha_2}F^{\gamma}_{\alpha_3\alpha_4]}V^{\beta_2\beta_3}_{\beta\gamma}+V^{\beta\beta_1}_{[\alpha_1\alpha_2}V^{\gamma(\beta_2}_{\alpha_3\alpha_4]}F^{\beta_3)}_{\beta\gamma}+\\
&V^{\beta\gamma}_{[\alpha_1\alpha_2}F^{\beta_1}_{\alpha_3]\beta}V^{\beta_2\beta_3}_{\alpha_4\gamma}-V^{\beta\gamma}_{[\alpha_1\alpha_2}F^{\beta_1}_{\alpha_4]\beta}V^{\beta_2\beta_3}_{\alpha_3\gamma}-V^{\beta\gamma}_{[\alpha_1\alpha_4}F^{\beta_1}_{\alpha_3]\beta}V^{\beta_2\beta_3}_{\alpha_2\gamma}-V^{\beta\gamma}_{[\alpha_2\alpha_3}F^{\beta_1}_{\alpha_4]\beta}V^{\beta_2\beta_3}_{\alpha_1\gamma}+\\
&F^{\beta}_{[\alpha_1\alpha_2}V^{\gamma\beta_1}_{\alpha_3]\beta}V^{\beta_2\beta_3}_{\alpha_4\gamma}-F^{\beta}_{[\alpha_1\alpha_2}V^{\gamma\beta_1}_{\alpha_4]\beta}V^{\beta_2\beta_3}_{\alpha_3\gamma}-F^{\beta}_{[\alpha_1\alpha_4}V^{\gamma\beta_1}_{\alpha_3]\beta}V^{\beta_2\beta_3}_{\alpha_2\gamma}-F^{\beta}_{[\alpha_2\alpha_3}V^{\gamma\beta_1}_{\alpha_4]\beta}V^{\beta_2\beta_3}_{\alpha_1\gamma}=0.
\end{aligned} \tag{B.4}$$

The summand in the second line of (B.4) is equal to zero because it can be presented as

$$\begin{aligned}
&V^{\beta\gamma}_{[\alpha_1\alpha_2}F^{\beta_1}_{\alpha_3]\beta}V^{\beta_2\beta_3}_{\alpha_4\gamma}-V^{\beta\gamma}_{[\alpha_1\alpha_2}F^{\beta_1}_{\alpha_4]\beta}V^{\beta_2\beta_3}_{\alpha_3\gamma}-V^{\beta\gamma}_{[\alpha_1\alpha_4}F^{\beta_1}_{\alpha_3]\beta}V^{\beta_2\beta_3}_{\alpha_2\gamma}-V^{\beta\gamma}_{[\alpha_2\alpha_3}F^{\beta_1}_{\alpha_4]\beta}V^{\beta_2\beta_3}_{\alpha_1\gamma}=\\
&F^{\beta_1}_{\alpha_1\beta}V^{\beta\gamma}_{[\alpha_2\alpha_3}V^{\beta_2\beta_3}_{\alpha_4]\gamma}+F^{\beta_1}_{\alpha_2\beta}V^{\beta\gamma}_{[\alpha_1\alpha_4}V^{\beta_2\beta_3}_{\alpha_3]\gamma}+F^{\beta_1}_{\alpha_3\beta}V^{\beta\gamma}_{[\alpha_1\alpha_2}V^{\beta_2\beta_3}_{\alpha_4]\gamma}+F^{\beta_1}_{\alpha_4\beta}V^{\beta\gamma}_{[\alpha_1\alpha_3}V^{\beta_2\beta_3}_{\alpha_2]\gamma}=0
\end{aligned} \tag{B.5}$$

due to (35). In its turn, taking into account (35), we obtain the following representation of the summand in the third line of (B.4)

$$\begin{aligned}
F^{\beta}_{[\alpha_1\alpha_2}V^{\gamma\beta_1}_{\alpha_3]\beta}V^{\beta_2\beta_3}_{\alpha_4\gamma}-F^{\beta}_{[\alpha_1\alpha_2}V^{\gamma\beta_1}_{\alpha_4]\beta}V^{\beta_2\beta_3}_{\alpha_3\gamma} \quad &- \quad F^{\beta}_{[\alpha_1\alpha_4}V^{\gamma\beta_1}_{\alpha_3]\beta}V^{\beta_2\beta_3}_{\alpha_2\gamma}-F^{\beta}_{[\alpha_2\alpha_3}V^{\gamma\beta_1}_{\alpha_4]\beta}V^{\beta_2\beta_3}_{\alpha_1\gamma}=\\
&= \quad F^{\beta}_{[\alpha_1\alpha_2}V^{\gamma\beta_1}_{\alpha_3\alpha_4]}V^{\beta_2\beta_3}_{\beta\gamma}.
\end{aligned} \tag{B.6}$$

Then, by virtue of the obvious symmetry properties

$$F^{\beta}_{[\alpha_1\alpha_2}V^{\gamma\beta_1}_{\alpha_3\alpha_4]}V^{\beta_2\beta_3}_{\beta\gamma}=V^{\gamma\beta_1}_{[\alpha_1\alpha_2}F^{\beta}_{\alpha_3\alpha_4}V^{\beta_2\beta_3}_{\beta\gamma}=-V^{\gamma\beta_1}_{[\alpha_1\alpha_2}F^{\beta}_{\alpha_3\alpha_4}V^{\beta_2\beta_3}_{\gamma\beta}, \tag{B.7}$$

from (B.4) we finally find

$$V^{\beta\beta_1}_{[\alpha_1\alpha_2}V^{\gamma(\beta_2}_{\alpha_3\alpha_4]}F^{\beta_3)}_{\beta\gamma}=0. \tag{B.8}$$

which is the first identities in (42).

To prove the second identities in (42), consider the Jacobi identities (10) written in the form

$$V^{\gamma(\beta_2}_{\alpha_3\alpha_4}V^{\beta_3\sigma)}_{\beta\gamma}+V^{\gamma(\beta_2}_{\alpha_4\beta}V^{\beta_3\sigma)}_{\alpha_3\gamma}+V^{\gamma(\beta_2}_{\beta\alpha_3}V^{\beta_3\sigma)}_{\alpha_4\gamma}=0. \tag{B.9}$$

Multiplying these identities by $V^{\beta\beta_1}_{\alpha_1\alpha_2}$ and summing over β we have

$$V^{\beta\beta_1}_{\alpha_1\alpha_2}V^{\gamma(\beta_2}_{\alpha_3\alpha_4}V^{\beta_3\sigma)}_{\beta\gamma}+V^{\beta\beta_1}_{\alpha_1\alpha_2}V^{\gamma(\beta_2}_{\alpha_4\beta}V^{\beta_3\sigma)}_{\alpha_3\gamma}+V^{\beta\beta_1}_{\alpha_1\alpha_2}V^{\gamma(\beta_2}_{\beta\alpha_3}V^{\beta_3\sigma)}_{\alpha_4\gamma}=0. \tag{B.10}$$

After antisymmetrizing in indices $\alpha_1,\alpha_2,\alpha_3,\alpha_4$ we derive

$$\begin{aligned}
V^{\beta\beta_1}_{[\alpha_1\alpha_2}V^{\gamma(\beta_2}_{\alpha_3\alpha_4]}V^{\beta_3\sigma)}_{\beta\gamma} \quad &= \quad V^{\beta\beta_1}_{[\alpha_1\alpha_2}V^{\gamma(\beta_2}_{\alpha_3]\beta}V^{\beta_3\sigma)}_{\alpha_4\gamma}-V^{\beta\beta_1}_{[\alpha_4\alpha_1}V^{\gamma(\beta_2}_{\alpha_2]\beta}V^{\beta_3\sigma)}_{\alpha_3\gamma}+\\
&\quad\;\; V^{\beta\beta_1}_{[\alpha_3\alpha_4}V^{\gamma(\beta_2}_{\alpha_1]\beta}V^{\beta_3\sigma)}_{\alpha_2\gamma}-V^{\beta\beta_1}_{[\alpha_2\alpha_3}V^{\gamma(\beta_2}_{\alpha_4]\beta}V^{\beta_3\sigma)}_{\alpha_1\gamma}.
\end{aligned} \tag{B.11}$$

Each term in rhs. of (B.11) is equal to zero because of (35). Therefore we obtain

$$V^{\beta\beta_1}_{[\alpha_1\alpha_2}V^{\gamma(\beta_2}_{\alpha_3\alpha_4]}V^{\beta_3\sigma)}_{\beta\gamma} = 0. \tag{B.12}$$

If we start with the relations (35) then in a similar way we can obtain the relations

$$V^{\beta\beta_1}_{[\alpha_1\alpha_2}V^{\gamma\beta_2}_{\alpha_3\alpha_4]}V^{\beta_3\sigma}_{\beta\gamma} = 0 \tag{B.13}$$

which were used in Section 2 to find structure functions $U^{(3)\beta_1\beta_2\beta_3}_{\alpha_1\alpha_2\alpha_3\alpha_4}$.

Now we consider the proof of identities (74) and (75). In the case when restrictions (68) are valid then the Jacobi identities (50) and (51) are reduced to

$$F_{[\alpha\beta}^{\ \ \rho}V^{(m)\beta_1\ldots\beta_m\beta_{m+1}}_{\lambda]\rho} + V^{(m)\rho(\beta_1\ldots\beta_m}_{[\alpha\beta}F_{\lambda]\rho}^{\ \ \beta_{m+1})} = 0, \quad m = 1, ..., n-1. \tag{B.14}$$

From (68) and (B.14) one can easy derive the following relations

$$\bar{V}^{\sigma\alpha}_{[\mu\nu}\bar{V}^{\beta\gamma}_{\lambda]\sigma} = 0, \quad \tilde{V}^{\sigma\alpha}_{[\mu\nu}\tilde{V}^{\beta\gamma}_{\lambda]\sigma} = 0, \quad \bar{V}^{\sigma\alpha}_{[\mu\nu}\tilde{V}^{\beta\gamma}_{\lambda]\sigma} = 0, \quad \tilde{V}^{\sigma\alpha}_{[\mu\nu}\bar{V}^{\beta\gamma}_{\lambda]\sigma} = 0 \tag{B.15}$$

and

$$F_{[\alpha\beta}^{\ \ \rho}\bar{V}^{\beta_1\beta_2}_{\lambda]\rho} + \bar{V}^{\rho(\beta_1}_{[\alpha\beta}F_{\lambda]\rho}^{\ \ \beta_2)} = 0, \quad F_{[\alpha\beta}^{\ \ \rho}\tilde{V}^{\beta_1\beta_2}_{\lambda]\rho} + \tilde{V}^{\rho(\beta_1}_{[\alpha\beta}F_{\lambda]\rho}^{\ \ \beta_2)} = 0 \tag{B.16}$$

respectively.

Taking into account the proof given in the beginning of this Appendix one can conclude that from (B.15) and (B.16) it follows that

$$\bar{V}^{\sigma\beta_1}_{[\alpha_1\alpha_2}\bar{V}^{\rho(\beta_2}_{\alpha_3\alpha_4]}F^{\beta_3)}_{\sigma\rho} = 0, \quad \tilde{V}^{\sigma\beta_1}_{[\alpha_1\alpha_2}\bar{V}^{\rho(\beta_2}_{\alpha_3\alpha_4]}F^{\beta_3)}_{\sigma\rho} = 0, \quad \bar{V}^{\sigma\beta_1}_{[\alpha_1\alpha_2}\tilde{V}^{\rho(\beta_2}_{\alpha_3\alpha_4]}F^{\beta_3)}_{\sigma\rho} = 0, \tag{B.17}$$

$$\bar{V}^{\sigma\beta_1}_{[\alpha_1\alpha_2}\bar{V}^{\rho\beta_2}_{\alpha_3\alpha_4]}\bar{V}^{\beta_3\lambda}_{\sigma\rho} = 0, \quad \tilde{V}^{\sigma\beta_1}_{[\alpha_1\alpha_2}\tilde{V}^{\rho\beta_2}_{\alpha_3\alpha_4]}\bar{V}^{\beta_3\lambda}_{\sigma\rho} = 0, \quad \bar{V}^{\sigma\beta_1}_{[\alpha_1\alpha_2}\tilde{V}^{\beta_2\rho}_{\alpha_3\alpha_4]}\bar{V}^{\beta_3\lambda}_{\sigma\rho} = 0. \tag{B.18}$$

which immediately lead to identities (74), (75) and (76).

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